Published by Institute of Physics Publishing for SISSA
Received: May 7, 2008
Accepted: May 22, 2008
Published: June 4, 2008

A holographic QCD and excited baryons from string theory

Jaemo Park[abd] **and Piljin Yi**[cd]

[a] *Department of Physics, Postech, Pohang 790-784, Korea*
[b] *Postech Center for Theoretical Physics (PCTP), Postech, Pohang 790-784, Korea*
[c] *School of Physics, Korea Institute for Advanced Study, Seoul 130-722, Korea*
[d] *Department of Physics, Stanford University, Stanford, CA 94305-4060, U.S.A.*
E-mail: jaemo@postech.ac.kr, piljin@kias.re.kr

Abstract: We study baryons of arbitrary isospin in a stringy holographic QCD model. In this D4-D8 holographic setting, the flavor symmetry is promoted to a gauge symmetry in the bulk, and produces, as KK modes of the gauge field, pions and spin one mesons of low energy QCD. Baryons of arbitrary isospins are represented as instanton solitons with isospin and spin quantum numbers locked, in a manner similar to the Skyrmion model. The soliton picture leads to a natural effective field theory of arbitrary baryons interacting with mesons. Couplings of baryons to axial mesons, including pions, are dominated in the large N_c limit by a direct coupling to the flavor field strength in five dimensions. We delineate the relevant couplings and determine their strengths. This work generalizes part of refs. [8, 10] to all excited baryons. Due to technical difficulties in introducing relativistic higher spin fields, we perform all computations in the nonrelativistic regime, which suffices for the leading N_c predictions.

Keywords: D-branes, QCD, Gauge-gravity correspondence.

JHEP06(2008)011

Contents

1. A holographic QCD

The model starts with a stack of N_c D4 branes compactified on a thermal circle. Because the fermions are given anti-periodic boundary condition, the massless part of the theory is pure $\mathrm{U}(N_c)$ Yang-Mills theory. Scalars would be also massless classically but due to the broken supersymmetry they would acquire mass perturbatively, whereas the gauge fields remain massless, protected by the gauge symmetry.

Let us first introduce notations for various spacetime coordinates and indices. The Minkowskii coordinates in which the QCD lives and in which the noncompact part of N_c D4 branes lives, will be denoted as

$$x^\mu, \quad \mu = 0, 1, 2, 3 \ ,$$

while the spatial coordinates will be labelled as

$$x^i, \quad i = 1,2,3 \qquad \text{or} \quad x^a, \quad a = 1,2,3\,.$$

We will be forced to mix a, b, c indices, usually reserved as SU(2) gauge indices, and the spatial i, j, k due to the spin-isospin mixing of the baryon. The holographic direction provides another spatial direction, whose coordinate will be either U or w. w is the particular choice, where the relevant five-dimensional geometry has a conformally flat coordinate (x^μ, w). Adding this fifth coordinate, we have

$$x^{\hat{M}}, \quad \hat{M} = 0,1,2,3,4 \qquad \text{or} \quad x^M, \quad M = 0,1,2,3,4\ ,$$

and

$$x^m, \quad m = 1,2,3,4\ ,$$

where $x^4 = w$. The hatted indices, as in $x^{\hat{M}}$, are raised and lowered using the proper induced (conformally flat) metric on the D-brane, whereas unhatted indices are raised and lowered using the flat metric. The rest of the stringy ten dimensions are spanned by S^4 and one angle, τ, which is the thermal circle wrapped by the N_c D4 branes.

In the large N_c limit, the dynamics of these D4 is dual to a closed string theory in some curved background with flux in accordance with the general AdS/CFT idea [1]. In the large 't Hooft coupling limit, $\lambda \equiv g_{\rm YM}^2 N_c \gg 1$, and neglecting the gravitational back-reaction from the D8 branes, the metric is [2]

$$ds^2 = \left(\frac{U}{R}\right)^{3/2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + f(U)d\tau^2\right) + \left(\frac{R}{U}\right)^{3/2}\left(\frac{dU^2}{f(U)} + U^2 d\Omega_4^2\right)\ , \tag{1.1}$$

with $R^3 = \pi g_s N_c l_s^3$ and $f(U) = 1 - (U_{KK}/U)^3$. The coordinate τ is compactified as $\tau = \tau + \delta\tau$ with $\delta\tau = 4\pi R^{3/2}/(3U_{KK}^{1/2})$. The lowest energy sector of this dual geometry encodes low energy theory of pure SU(N_c) Yang-Mills theory. Glueball spectrum from this dual setup has been computed with some successful predictions against lattice results [3, 4].

To add mesons, we introduce the N_F D8 branes sharing the coordinates x^0, x^1, x^2, x^3 with the D4 branes [5]. This allows massless quark degrees of freedom as open strings attached to both the D4 and D8 branes. As the D4's are replaced by the geometry, however, the 4-8 open strings are paired into 8-8 open strings, which are naturally identified as bi-quark mesons. From the viewpoint of D8 branes, these mesons arise out of a U(N_F) Yang-Mills theory with the extra Chern-Simons coupling,

$$\mu_8 \int \sum C_{p+1} \wedge {\rm Tr}\, e^{2\pi\alpha' \mathcal{F}}\ , \tag{1.2}$$

on the D8 branes. We defined $\mu_p = 2\pi/(2\pi l_s)^{p+1}$, and $l_s^2 = \alpha'$. C_{p+1}'s are the antisymmetric Ramond-Ramond fields.

The induced metric on the D8 brane is

$$g_{8+1} = \left(\frac{U}{R}\right)^{3/2}\left(\eta_{\mu\nu}dx^\mu dx^\nu\right) + \left(\frac{R}{U}\right)^{3/2}\left(\frac{dU^2}{f(U)} + U^2 d\Omega_4^2\right)\ . \tag{1.3}$$

A useful choice of the coordinate is[1]

$$w = \int_{U_{KK}}^{U} \frac{R^{3/2} dU'}{\sqrt{U'^3 - U_{KK}^3}} \,. \tag{1.5}$$

with which we have

$$g_{8+1} = \frac{U^{3/2}(w)}{R^{3/2}} \left(dw^2 + \eta_{\mu\nu} dx^\mu dx^\nu \right) + \frac{R^{3/2}}{U^{1/2}(w)} d\Omega_4^2 \,. \tag{1.6}$$

The noncompact part of the D8 brane worldvolume is conformally equivalent to an interval $[-w_{\rm max}, w_{\rm max}]$ times R^{3+1} with

$$w_{\rm max} = \int_0^\infty \frac{R^{3/2} dU}{\sqrt{U^3 - U_{KK}^3}} = \frac{1}{M_{KK}} \frac{3}{2} \int_1^\infty \frac{d\tilde{U}}{\sqrt{\tilde{U}^3 - 1}} \tag{1.7}$$

which makes the search for exact instanton solution rather problematic.

Let us list parameters of the background. We have

$$R^3 = \frac{g_{\rm YM}^2 N_c l_s^2}{2 M_{KK}} \,, \qquad U_{KK} = \frac{2 g_{\rm YM}^2 N_c M_{KK} l_s^2}{9} \,, \tag{1.8}$$

so that $M_{KK} \equiv 3U_{KK}^{1/2}/2R^{3/2}$. Also the nominal Yang-Mills coupling $g_{\rm YM}^2$ is related to the other parameters as

$$g_{\rm YM}^2 = 2\pi g_s M_{KK} l_s \,. \tag{1.9}$$

where g_s is the string coupling, but is not a physical parameter on its own. The low energy parameters of this holographic theory are M_{KK} and λ, which together with N_c sets all the physical scales such as the QCD scale and the pion decay constant. Another important quantity to have in mind is

$$l_s^{\rm warped} \equiv l_s \times (R/U_{KK})^{3/4} \simeq \frac{2.6}{M_{KK}\sqrt{\lambda}} \,, \tag{1.10}$$

which is basically the warped string length scale. This is the string length scale as measured by x^μ coordinates at $U = U_{KK}$.

In the low energy limit, the worldvolume dynamics of the D8 brane is described in terms of a derivative expansion of the full stringy effective action. The effective action is

$$-\frac{1}{4} \int_{4+1} \sqrt{-g_{4+1}} \, \frac{e^{-\Phi} V_{S^4}}{2\pi (2\pi l_s)^5} \, {\rm tr} \mathcal{F}_{\hat{M}\hat{N}} \mathcal{F}^{\hat{M}\hat{N}} + \frac{N_c}{24\pi^2} \int_{4+1} \omega_5(\mathcal{A}) \,, \tag{1.11}$$

with $d\omega_5(\mathcal{A}) = {\rm tr} \mathcal{F}^3$. Here V_{S^4} is the position-dependent volume of the compact part with

$$V_{S^4} = \frac{8\pi^2}{3} R^3 U \,, \tag{1.12}$$

[1]Another choice of the radial coordinate z defined as

$$U^3 = U_{KK}^3 + U_{KK} z^2 \,, \tag{1.4}$$

was used by Sakai and Sugimoto [5]. Near origin $w \simeq 0$, we have $M_{KK} w \simeq z/U_{KK}$.

while the dilaton is

$$e^{-\Phi} = \frac{1}{g_s}\left(\frac{R}{U}\right)^{3/4} . \tag{1.13}$$

The Chern-Simons coupling arises from the second set of terms because $\int_{S^4} dC_3 \sim N_c$ takes a quantized value, and was worked out by Sakai and Sugimoto in some detail [5].

The massless sector upon dimensional reduction to four dimension produces the Chiral lagrangian with a Skyrme term [6]. The pion field π is conveniently expressed in the exponentiated forms

$$\mathrm{U}(x) = e^{2i\pi(x)/f_\pi} \,, \xi(x) = e^{i\pi(x)/f_\pi} \,, \tag{1.14}$$

which can be found in the five-dimensional gauge field in the following expansion, with the gauge choice $\mathcal{A}_w = 0$,

$$\mathcal{A}_\mu(x;w) = i\alpha_\mu(x)\psi_0(w) + i\beta_\mu(x) + \sum_n a_\mu^{(n)}(x)\psi_{(n)}(w) \,, \tag{1.15}$$

where the SU(N_F) part of the lowest lying modes are directly connected to the pion field as

$$\alpha_\mu(x)^{\mathrm{SU}(N_F)} \equiv \left\{\xi^{-1}, \partial_\mu\xi\right\} \simeq \frac{2i}{f_\pi}\partial_\mu\pi, \qquad \beta_\mu(x)^{\mathrm{SU}(N_F)} \equiv \frac{1}{2}\left[\xi^{-1}, \partial_\mu\xi\right] \simeq \frac{1}{2f_\pi^2}[\pi, \partial_\mu\pi] \,, \tag{1.16}$$

where $\psi_0(w) = \psi_0(w(z)) = \frac{1}{\pi}\arctan\left(\frac{z}{U_{KK}}\right)$. Truncating to pions only, this reproduces the Skyrme Lagrangian [6]

$$\mathcal{L}_{\mathrm{pion}} = \frac{f_\pi^2}{4}\mathrm{tr}\left(U^{-1}\partial_\mu U\right)^2 + \frac{1}{32e^2_{Skyrme}}\mathrm{tr}\left[U^{-1}\partial_\mu U, U^{-1}\partial_\nu U\right]^2 \,, \tag{1.17}$$

with

$$f_\pi^2 = \frac{1}{54\pi^4}(g_{\mathrm{YM}}^2 N_c)N_c M_{KK}^2 \,, \quad e^2_{\mathrm{Skyrme}} \simeq \frac{54\pi^7}{61}\frac{1}{(g_{\mathrm{YM}}^2 N_c)N_c} \,. \tag{1.18}$$

For the rest of KK tower, which are vector mesons and axial vector mesons, we have the standard kinetic term

$$\mathcal{L}^{\mathrm{free}}_{\mathrm{vectors}} = \sum_n \left\{\frac{1}{4}\mathcal{F}^{(n)}_{\mu\nu}\mathcal{F}^{\mu\nu(n)} + \frac{1}{2}m_n^2 a_\mu^{(n)} a^{\mu(n)}\right\} \,, \tag{1.19}$$

with $\mathcal{F}^{(n)}_{\mu\nu} = \partial_\mu a_\nu^{(n)} - \partial_\nu a_\mu^{(n)}$, plus various interactions between them as well as with pions. Finally there is the WZW term $\mathcal{L}_{WZW}$ also, arising from the Chern-Simons term, details of which can be found in [5].

One should in principle include other hadrons of QCD to this picture to have a complete QCD-like theory. Glueballs are already present in the setup as the gravity part of the holographic theory but a systematic study of glueball/meson interaction is not available beyond the initial but interesting study in ref. [7]. On the other hand, experimentally, proper identification of glueballs is not available so comparison with data is not easy. The other, obvious, set of hadrons are baryons whose properties have been explored recently [8 – 10]. In this work, we wish to generalize and expand these recent studies to baryons of arbitrary isospins.[2]

[2]Another interesting direction of study involving holographic baryon in Sakai-Sugimoto model is to consider physics in the background of a baryon density. See for example recent works in ref. [11 – 13].

2. A note on the effective field theory approach

We reviewed above how the low energy effective theory of mesons emerges from this holographic setup. Before we go into the discussion of baryons, it is worthwhile to clarify how we are meant to use the effective action thus derived. The AdS/CFT correspondence in general is meant to be a conjectured duality between an open string theory and a closed string theory. As such, we anticipate such a correspondence at full quantum level on both sides. In practice, however, we often must resort to large N_c and large 't Hooft coupling limit where we at least can compute quantities on the closed string side. This usual limit allows us to treat the closed string side as a classical theory of gravity and its multiplets [1].

Approaches to holographic QCD so far have not escaped this limitation. As a result, when we consider the same large N_c and large λ limit, the so-called "bulk side" is meant to be used classically. The effective field theory such as above is derived strictly in this spirit, and is meant to be used classically. In other words, we should not try to renormalize it further by computing loop diagrams. We are only allowed to compute tree-level amplitudes using such the vertices present in the effective action. In this sense, the effective action here is an one-particle irreducible action (1PI) with all physical excitations already incorporated, rather than a Wilsonian effective action with a cut-off scale.

This statement has a caveat in the case of Sakai-Sugimoto type models where the mesons are introduced as degrees of freedom on a probe brane. What the latter means is that the loop effects of the quark-like particle are not taken into account by this holographic prescription. In other words, such a holographic model will at best match with quenched version of QCD. This is to be expected when N_c is large and N_F is finite, since the fermion loops would be suppressed by N_F/N_c. It is only when one tries to extrapolate to the real QCD regime of $N_c = 3$ that we must worry about how quenching of the fermion should be counteracted. However, in this paper, we will work within the spirit of large N_c QCD, and ignore this issue.

As was studied in depth recently [8, 9], the baryon appears in an entirely different manner. One may recall that, in the conventional Chiral lagrangian approach, the baryon appears as a nonperturbative soliton called Skyrmion. In this holographic and five-dimensional setup, Skyrmion is replaced by another type of soliton which carries unit Pontryagin number in the bulk. We will call it an instanton soliton. Furthermore, the instanton soliton has been shown to shrink to a size $\sim 1/(M_{KK}\sqrt{\lambda})$ and be localized at the center of the fifth direction. An important advantage in the small soliton size is that one naturally can resort to an effective field theory language in the precisely the same sense as the above effective action of mesons.

For large solitons, which are semi-classical objects, introduction of an effective field could be a tricky business since we would be trying to represent a large fluffy objects in terms of point-like quanta of an effective field. On the other hand, we know from study of dualities that sometimes one can formulate a theory with soliton in terms of a new field whose elementary excitation is identified with the soliton. When would it be justified? It is justified precisely when the parameters of the original theory approaches a strong coupling regime so that the size of the soliton becomes smaller that the typical length scales of the

JHEP06(2008)011

theory.

In this example, we have a soliton whose Compton size scales as $1/(\lambda N_c M_{KK})$ and whose soliton size scales as $1/(\sqrt{\lambda} M_{KK})$. In contrast, the mass scales of the mesons are fixed at M_{KK}. Thus, both the Compton size and the soliton size of the baryon is much smaller than any of the meson scale. This tells us is that it is perfectly sensible to introduce an effective field in place of the soliton for the purpose of studying interactions with the meson sector. On another side of the matter, the classical soliton picture remains robust since its Compton size is smaller than the soliton size, which allows us to exploit properties of the classical soliton solution (whose classical field is made out of mesons) in reading out interactions of the mesons with the soliton.

In refs. [8, 10, 14], this program was carried out for the lowest lying excitation of the soliton, to be identified with the nucleons with are of isospin 1/2. However, there is really no reason to truncate to nucleons since the next excitation, say isospin 3/2 Δ particles, are not too heavy compared to the nucleons. The purpose of this note is to extend this program and read out baryons of arbitrary isospin and their interactions with mesons and other baryons.

3. Baryons as 5D solitons

A wrapped D4 brane along the compact S^4 corresponds to a baryon vertex on the five-dimensional spacetime [5], which follows from an argument originally given by Witten [15]. To distinguish such D4 from QCD D4's, let us call them D4'. On their worldvolume brane we have a Chern-Simons coupling of the form,

$$\mu_4 \int C_3 \wedge 2\pi\alpha' d\tilde{\mathcal{A}} = 2\pi\alpha'\mu_4 \int dC_3 \wedge \tilde{\mathcal{A}} \tag{3.1}$$

with D4' gauge field $\tilde{\mathcal{A}}$. Since D4' wraps the S^4 which has a quantized N_c flux of dC_3, one finds that this term induces N_c unit of electric charge on the wrapped D4'. The Gauss constraint for $\tilde{\mathcal{A}}$ demands that the net charge should be zero, however, and the D4' can exist only if N_c fundamental strings end on it. In turn, the other end of fundamental strings must go somewhere, and the only place it can go is D8 branes. Thus a D4' wrapping S^4 looks like an object with electric charge with respect to the gauge field on D8. With respect to the overall U(1) of the latter, which counts the baryon number, the charge is N_c. Thus, we may identify the baryon as wrapped D4' with N_c fundamental strings sticking onto it.

3.1 Baryons as "small" instantons with Coulombic hair

This wrapped D4' dissolves into D8 branes and become an instanton soliton on the latter.[3]

[3]Usual low energy QCD picture of baryons as the Skyrmion is directly related to this instanton picture. As was pointed out by Atiyah and Manton [16], an open Wilson line in the presence of an instanton carries the Skyrmion winding number. Here, the Wilson line along the holographic direction is nothing but the pion field U, completing this correspondence between the instanton picture and the Skyrmion picture. From this viewpoint, the instanton soliton can be thought of as the Skyrmion which is corrected by the infinite tower of vector and axial vector mesons. Corrections after including the lightest vector meson only has been previously considered in ref. [17, 18]. However, the full holographic picture seems to change the large N_c nature soliton more profoundly.

JHEP06(2008)011

The reason for why D4' cannot dissociate away from D8 is obvious. The D4' has N_c fundamental strings attached, whose other ends are tied to D8. Moving away from D8 by distance L means acquiring extra mass of order $N_c L/l_s^2$ due to the increased length of the strings, so the D4' would have to stay on top of D8 for a simple energetics reason [8, 9].

Once on top of D8's, the D4' is replaced by an instanton configuration with

$$\frac{1}{8\pi^2}\int_{R^3\times I} \mathrm{tr} F\wedge F = 1\,, \tag{3.2}$$

where F is the SU(N_F) part of the D8 gauge field strength $\mathcal{F}$. This is a well-known consequence of the Chern-Simons term on D8,

$$\mu_8 \int_{R^{3+1}\times I\times S^4} C_5\wedge 2\pi^2(\alpha')^2 \mathrm{tr} F\wedge F = \mu_4\int_{R^{0+1}\times S^4} C_5\wedge \frac{1}{8\pi^2}\int_{R^3\times I}\mathrm{tr} F\wedge F\,, \tag{3.3}$$

which shows that a unit instanton couples to C_5 minimally, and carries exactly one unit of D4' charge.

How about the size of the instanton soliton? Consider the kinetic part of D8 brane action, compactified on S^4, in the Yang-Mills approximation,

$$-\frac{1}{4}\int \sqrt{-g_{4+1}}\,\frac{e^{-\Phi}V_{S^4}}{2\pi(2\pi l_s)^5}\,\mathrm{tr}\mathcal{F}_{\hat M\hat N}\mathcal{F}^{\hat M\hat N} = -\int dx^4 dw\,\frac{1}{4e^2(w)}\,\mathrm{tr}\mathcal{F}_{MN}\mathcal{F}^{MN}\,, \tag{3.4}$$

where the unhatted indices are those associated with the flat metric $dx_\mu dx^\mu + dw^2$, and the electric coupling is w-dependent,

$$\frac{1}{e^2(w)} \equiv \frac{8\pi^2 R^3\,\mathrm{U}(w)}{3(2\pi l_s)^5(2\pi g_s)} = \frac{(g_{\rm YM}^2 N_c)N_c}{108\pi^3}M_{KK}\frac{\mathrm{U}(w)}{U_{KK}}\,. \tag{3.5}$$

Suppose that we have a point-like instanton localized at $w=0$. Its energy from the Yang-Mills kinetic term would be the standard instanton action,

$$m_B^{(0)} \equiv \frac{4\pi^2}{e^2(0)} = \frac{(g_{\rm YM}^2 N_c)N_c}{27\pi}M_{KK}\,. \tag{3.6}$$

For a slightly larger instanton, on the other hand, w-dependence of $e(w)^2$ will induce more energy since the kinetic term is proportional to $1/e(w)^2$. For small size parameter ρ such that $\rho M_{KK}\ll 1$, this extra energy is[4]

$$\delta m_B^{\rm Pontryagin} \simeq \frac{1}{6}\,m_B^{(0)}M_{KK}^2\rho^2\,, \tag{3.7}$$

Thus, in the absence of any other effect, the instanton would shrink to $\rho=0$.

On the other hand, the instanton soliton is really a representation of a wrapped D4' with N_c fundamental strings attached. The effect of these fundamental strings are encoded in the world-volume gauge theory as a Chern-Simons term,

$$\frac{N_c}{24\pi^2}\int \omega_5(\mathcal{A})\,, \tag{3.8}$$

[4] The estimate of energy here takes into account the spread of the instanton density $D(x^i,w)\sim \rho^4/(r^2+w^2+\rho^2)^4$, but ignores the deviation from the flat geometry along the four spatial directions.

which implies that, for $N_F = 2$, the U(1) part of $\mathcal{A}$ will see a charge density proportional to the Pontryagin density of the instanton. Since the Coulomb repulsion favors less and less dense charge distribution, this effect goes to expand the instanton size. More precisely, the five dimensional Coulomb energy goes as

$$\delta m_B^{\rm Coulomb} \simeq \frac{1}{2} \times \frac{e(0)^2 N_c^2}{10\pi^2\rho^2} \,, \tag{3.9}$$

again provided that $\rho M_{KK} \ll 1$.

The competition of the two effects sets the size to minimize $\delta m_B^{\rm Coulomb} + \delta m_B^{\rm Pontryagin}$, which is achieved at [8, 9]

$$\rho_{\rm baryon} \simeq \frac{(2 \cdot 3^7 \cdot \pi^2/5)^{1/4}}{M_{KK}\sqrt{\lambda}} \,, \tag{3.10}$$

with the classical mass

$$m_B^{\rm classical} = m_B^{(0)} \times \left(1 + \frac{\sqrt{2 \cdot 3^5 \cdot \pi^2/5}}{\lambda} + \cdots \right). \tag{3.11}$$

For an arbitrarily large 't Hooft coupling limit, the size $\rho_{\rm baryon}$ is significantly smaller than the Compton sizes of the mesons $\sim 1/M_{KK}$ but much larger than its own Compton size $1/m_B^{\rm classical} \simeq 27\pi/(M_{KK}\lambda N_c)$.

Before proceeding further, we should point out that the size of the soliton scales the same way as $l_s^{\rm warped}$. This tells us that the Yang-Mills Chern-Simons action we used so far may not be completely reliable. Plugging in the numbers, we see that the size of the soliton is about four times larger than $l_s^{\rm warped}$, making it not too small but not large enough to avoid stringy corrections either. Consideration of higher order stringy effects will likely shift the size estimate we have here, making a quantitative correction. Whether or not we should include these corrections depends on what we wish to do. It is true that the stringy theory model at hand clearly demands any such corrections be included. On the other hand, the stringy holographic QCD model should be reliably dual to ordinary QCD only in the energy scale far below M_{KK} anyway, yet have successfully reproduced certain behaviors of QCD around M_{KK} as well. How and why of this are hardly clear for this model, nor is it clear for any other holographic QCD. In this sense, the guiding principle is lost once we begin to consider any massive objects, as far as we are interested in emulating real QCD.

With this uncertainty in mind, we will try to proceed without worrying about such stringy corrections. A good news is, though, that, in what follows from here, where we effectively consider soft scattering processes involving meson, this problems is much less acute. Even though the mass scale and the length scale of the hadrons are dangerously high and small, actual physical process to be considered are such that the momentum transfer is typically no larger than M_{KK} and more like f_π. When we compute corrections to $\rho_{\rm baryon}$ by whatever higher order effect, all we have to do is to replace our size parameter by the corrected one in what follows, and the rest is intact.

3.2 Quantization

If the soliton size is small, physics near a soliton located at $w = 0$ retains the approximate symmetry of R^{4+1}. The deviation from this symmetry is an important ingredient that enters the size estimate of the soliton and also must be considered carefully for reading out interactions between baryons and mesons. However, we will temporarily ignore this deviation since here we concentrate on the counting of the quantum states, for which the approximate R^{4+1} Minkowskian invariance can be very useful, and the result robust under the deviations. Matching of quantized soliton with the baryon is easiest when the number of flavor is two. From this point on, we specialize to the case of $N_F = 2$.

In order to set up an effective action of baryons, it is important to understand what kind of quantum states emerges from quantizing these solitons. Usual SU(2) instanton in flat R^4 carries eight collective coordinates, four translational ones, three global SU(2) rotations, and one overall size. Of these, the last is not a moduli direction for our instanton, but the other seven are all from broken symmetry and thus remain flat. To elevate the instanton soliton to a point-like object, i.e. a quantum of an effective field, we must quantize some of these collective coordinates and produces representations under the symmetry of the moduli space.

The approximate Lorentz group at hand is SO(4, 1), to be broken to SO(3, 1) by the curvature effect etc. The approximate little group for massive particle is $\mathrm{SO}(4)_{R^4} = \mathrm{SU}(2)_+ \times \mathrm{SU}(2)_-$. Classical self-dual instanton rotates nontrivially under one of the two factors, say $\mathrm{SU}(2)_+$, while classical anti-instanton rotates under $\mathrm{SU}(2)_-$. Instantons also gets rotated by the global gauge rotation $\mathrm{SU}(N_F = 2)$,

$$F \quad \rightarrow \quad S^\dagger F S \, , \tag{3.12}$$

with special unitary matrices S. The collection of S spans the SU(2) manifold, or equivalently $\mathbf{S}^3$, but since S and $-S$ rotates the solution the same way the moduli space is naively $\mathbf{S}^3/Z_2$. However at quantum level, we must consider states odd under this Z_2 as well, so the moduli space is $\mathbf{S}^3$. Then, its quantization is a matter of finding eigenstates of free and nonrelativistic nonlinear sigma-model onto $\mathbf{S}^3$ [20 – 22].

S itself admits SO(4) symmetry action of its own, which can be written as

$$S \quad \rightarrow \quad U S V^\dagger \, . \tag{3.13}$$

Because of the way the spatial indices are locked with the gauge indices, these two rotations are each identified as the gauge rotation, $\mathrm{SU}(2)_I$, and half of spatial rotation, $\mathrm{SU}(2)_+$. For each factor, we have a triplet of symmetry operators, $I_{1,2,3}$ and $J^{(+)}_{1,2,3}$, respectively.

Eigenstates on $\mathbf{S}^3$ are nothing but the angular momentum eigenstates under I's and $J^{(+)}$'s, conventionally denoted as

$$|s : p, q\rangle \, , \tag{3.14}$$

with the eigenvalues $I^2 = s(s + 1) = (J^{(+)})^2$, $I_3 = p$, and $J^{(+)}_3 = q$. As is well-known, I^2 and J^2 eigenvalues are always equal on $\mathbf{S}^3$, so that spin s baryons are always in isospin s representation as well. For even s, $D^{(s)}$'s are even under the Z_2, and for odd s, $D^{(s)}$'s are odd under Z_2.

The simplest way to represent these eigenstates as wavefunctions on $\mathbf{S}^3$ are to use the Cartesian representation of the Euler angles as

$$S = S(\xi) = \xi_4 + i\xi_a\tau_a \, , \quad \xi^2 = 1 \, , \tag{3.15}$$

for the 2×2 Pauli matrices τ_i's. The eigenstates have a well-known representation in the coordinate basis as functions on $\mathbf{S}^3$,

$$D^{(s)}_{pq}(\xi) = \langle \xi | s : p, q \rangle \ , \tag{3.16}$$

and we can further choose the basis for ξ's such that

$$D^{(s)}_{ss}(\xi) = \sqrt{\frac{2s+1}{2}} \, \frac{1}{\pi} \, (\xi_1 + i\xi_2)^{2s} \ . \tag{3.17}$$

The spin and isospin operators are realized as differential operators

$$\begin{aligned} I_a \to \mathcal{I}_a &\equiv -\frac{i}{2} \left(\epsilon_{abc} \xi_b \partial_c - \xi_4 \partial_a + \xi_a \partial_4 \right) \ , \\ J^{(+)}_a \to \mathcal{J}^{(+)}_a &\equiv -\frac{i}{2} \left(\epsilon_{abc} \xi_b \partial_c + \xi_4 \partial_a - \xi_a \partial_4 \right) \ . \end{aligned} \tag{3.18}$$

It is easy to show that $\mathcal{I}^2 = (\mathcal{J}^{(+)})^2$ holds as the consistency would require.

One can proceed exactly the same manner for anti-instantons, where $\mathrm{SU}(2)_+$ is replaced by $\mathrm{SU}(2)_-$. Therefore, under $\mathrm{SU}(2)_I \times \mathrm{SO}(4)_{R^4} = \mathrm{SU}(2)_I \times \mathrm{SU}(2)_+ \times \mathrm{SU}(2)_-$, quantized instantons are in

$$(2s+1; 2s+1; 1) \tag{3.19}$$

while quantized anti-instantons are in

$$(2s+1; 1; 2s+1). \tag{3.20}$$

Theoretically possible values for s are integers or half-integers. However, we are mainly interested in fermionic baryons, and will subsequently consider the case of half-integral s's.[5]

Before closing, let us note that the instanton and anti-instanton can be naturally thought of as particle/anti-particle pairs. The representation under the little group reflects this as well. Later, we will introduce an effective field whose elementary quanta are these particles and anti-particles. Due to CPT, the particle and the anti-particle always come together, and we expect to find an effective field that produce excitations that belong to

$$(2s+1)_{\mathrm{SU}(2)_I} \otimes \left((2s+1; 1) \oplus (1; 2s+1) \right)_{\mathrm{SO}(4)_{R^4}} \tag{3.21}$$

on-shell. This sets the table for extracting quantum baryons out of the classical instanton soliton.

4. Effective field theory of the instanton soliton

In this section, we wish to introduce an effective field whose elementary quanta are the quantized baryons of the previous section. We introduce the field content for any given isospin and propose an effective action of such baryon fields interacting with the gauge field, a.k.a., mesons. In next section, we will derive the proposed effective action by generalizing a method originally due to Adkins, Nappi, and Witten [20] and also adapted for holographic Nucleons in refs. [8, 10].

[5] For N_F larger than two, the half-integral spin of the baryons should follows immediately whenever N_c is odd, in a manner similar to the Skyrmion case [21].

4.1 Higher spin fields in five dimensions

We learned that quantization of instanton soliton leads to quantum states with isospin and spin related. For isospin s, the quantized instantons are in $(2s+1,1)$ and the quantized anti-instantons are in $(1,2s+1)$ under the little group $\mathrm{SO}(4) = \mathrm{SU}(2)_+ \times \mathrm{SU}(2)_-$. For $s = 1/2$, they combine into a Dirac field with a single spinor index, so one might think that, for $s = 3/2$, the relevant field is the Rarita-Schwinger field. However, this would be true only if we are working in four dimensions. The Rarita-Schwinger field in five dimensions produces six particle and six anti-particle degrees of freedom. More precisely they are in the representations $(3,2)+(2,3)$ under the little group.

The right choice is the higher spin field with multiple spin indices, completely symmetrized,

$$\Psi_{A_1A_2\cdots A_{2s}} = \Psi_{(A_1A_2\cdots A_{2s})} \; , \tag{4.1}$$

where the spin index A runs from 1 to 4. We consider half-integral s, since real QCD admits only those. Consider a free equation of motion

$$\gamma^M_{A_1B}\partial_M\Psi_{BA_2\cdots A_{2s}} = m\Psi_{A_1A_2\cdots A_{2s}} \; , \tag{4.2}$$

where the Dirac matrices act on the first spin index only. The Dirac operator squares to ∇^2, so we find the equation of motion implies the usual on-shell condition $p^2+m^2=0$, which of course gives $E^2=m^2$ in the rest frame. This further imposes the condition on the plane-wave spinors in the rest frame as

$$\mp i\gamma^0\Psi = \Psi \tag{4.3}$$

so that particles and anti-particles correspond to $-i\gamma^0$ eigenstates with eigenvalues ± 1. Due to the symmetrized spin indices, this implies that a $-i\gamma^0$ eigenstate must have the same "chirality" for all $2s$ indices.

On the other hand, $\Gamma \equiv \gamma^1\gamma^2\gamma^3\gamma^4 = -i\gamma^0$, so particles and anti-particles are, respectively, chiral and anti-chiral under the little group $\mathrm{SO}(4)_{R^4} = \mathrm{SU}(2)_+ \times \mathrm{SU}(2)_-$. With the following choice for Dirac matrices for $m,n = 0,1,2,3,4$,

$$\gamma^0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \tag{4.4}$$

particles are encoded in

$$\Psi_{(A_1A_2\cdots A_{2s})} \; , \quad A_i = 1,2 \; , \tag{4.5}$$

whose individual spinor indices $A_i = 1,2$ belong to doublets under $\mathrm{SU}(2)_+$. These particles are clearly in the spin s representation of $\mathrm{SU}(2)_+$. Likewise, anti-particles

$$\Psi_{(A_1A_2\cdots A_{2s})} \; , \quad A_i = 3,4 \; , \tag{4.6}$$

are in the spin s representation under $\mathrm{SU}(2)_-$.

The upshot is that the propagating degrees of freedom form

$$((2s+1,1)\oplus(1,2s+1)) \; , \tag{4.7}$$

under the little group $SO(4)_{R^4}$. After elevated to the isospin s under $SU(2)_I$, this spinor fields is then capable of reproducing particle contents of quantized instantons and anti-instantons. Since the spin and the isospin are locked, the field representing the quantized instanton and anti-instanton also carry the flavor SU(2) indices

$$\Psi^{\epsilon_1\cdots\epsilon_{2s}}, \quad \epsilon_i = 1, 2 \,. \tag{4.8}$$

For the case of $s = 1/2$, the authors of [8, 10] wrote down a relativistic field theory involving the nucleons and the gauge field.

For $s \geq 3/2$, however, this is easier said than done. For four dimensions, Rarita-Schwinger field does the trick for $s = 3/2$ but we cannot use this in five dimensions due to a different spin content. The only sensible way out, at least until we know better, is to employ the nonrelativistic approximation. This is well justified in the large λN_c limit, since the mass of the instanton scales as λN. Thus, instead of working with fully relativistic four-component spinor notations, we will split it to particle and anti-particles as

$$\begin{aligned} -i\gamma^0\Psi^{\epsilon_1\cdots\epsilon_{2s}} &= \Psi^{\epsilon_1\cdots\epsilon_{2s}} \to \mathcal{U}^{\epsilon_1\cdots\epsilon_{2s}}_{\alpha_1\cdots\alpha_{2s}}, \alpha_i = 1, 2 \,, \\ i\gamma^0\Psi^{\epsilon_1\cdots\epsilon_{2s}} &= \Psi^{\epsilon_1\cdots\epsilon_{2s}} \to \mathcal{V}^{\epsilon_1\cdots\epsilon_{2s}}_{\dot\beta_1\cdots\dot\beta_{2s}}, \dot\beta_i = 1, 2 \,. \end{aligned} \tag{4.9}$$

4.2 Nonrelativistic Lagrangian

In order to see the interaction terms giving rise to instanton tails it is sufficient to look for the nonrelativistic limit of higher spin theories. We just have to look for the interactions of the particle rather than antiparticle. As above, we denote

$$\mathcal{U}^{\epsilon_1\cdots\epsilon_{2s}}_{\alpha_1\cdots\alpha_{2s}} \tag{4.10}$$

be the positive energy components or particle components where a^i's, α^i's take values of 1,2. Indices a^i's, α^i's are all symmetrized. The minimal Lagrangian of the nonrelativistic limit of the particle of spin s is given by the usual Schrodinger type

$$S_0 = \int dtd^4x \,\left[\sum_s \left(i\mathcal{U}^\dagger_s \frac{\partial}{\partial t}\mathcal{U}_s + \sum_{m=1}^4 \frac{1}{2m(s)}\mathcal{U}^\dagger_s(\partial_m - i\mathcal{A}_m)^2\mathcal{U}_s \right)\right] \,, \tag{4.11}$$

where $\mathcal{U}_s$ denotes a field with spin s. Mass of the isospin s baryon is denoted as $m(s)$. Please see Hata et.al. [9] for an explicit formula of excited baryon mass. In the kinetic term, the U(2) gauge field enters in the following combination

$$\mathcal{A}_m = N_c A_{U(1)} + A^{(s)} \,, \tag{4.12}$$

where $A^{(s)}$ is the isospin s representation of the non-Abelian SU(2) part of the gauge field.

On the other hand, we anticipate additional couplings to the SU(2) field strength in much the same way for $s = 1/2$. The logic goes as follows. The above minimal interaction tells the gauge field to generate long-range Coulomb field in response to the electric charges on the soliton. However, the instanton and anti-instanton are characterized by the self-dual and anti-self-dual magnetic fields whose power-like tail is determined by ρ^2_{baryon}. Note that this magnetic field goes as of $1/r^4$, one more power of $1/r$ than the Coulomb field. When we replace the quantized instanton by a field, we must somehow incorporate this aspect of the soliton to re-emerge from the equation of motion, just as the Coulomb field emerges naturally from the minimal coupling. For the case of $s = 1/2$, it was shown in [8, 10] that a direct coupling to the field strength $F = dA + iA^2$ to a bilinear of the spinor emulates this long range behavior of the quantized soliton. We wish to generalize this to arbitrary s.

The proposal for these additional interaction terms are roughly

$$(\mathcal{U}^{\epsilon'\epsilon_2\cdots\epsilon_{2s}}_{\beta\alpha_2\cdots\alpha_{2s}})^*(\gamma^0\gamma^{KN})^{\beta\beta'}F^{\epsilon'\epsilon}_{KN}\,\mathcal{U}^{\epsilon\epsilon_2\cdots\epsilon_{2s}}_{\beta'\alpha_2\cdots\alpha_{2s}}\;, \tag{4.13}$$

between baryons of the same isospin, and

$$(\mathcal{U}^{\epsilon_1\epsilon_2\cdots\epsilon_{2s}}_{\alpha_1\alpha_2\cdots\alpha_{2s}})^*(\gamma^0 C\gamma^{KN})^{\beta\beta'}(\tau_2 F_{KN})^{\epsilon\epsilon'}\,\mathcal{U}^{\epsilon\epsilon'\epsilon_1\cdots\epsilon_{2s}}_{\beta\beta'\alpha_1\cdots\alpha_{2s}}\;, \tag{4.14}$$

between baryons of different isospins. Here, the spinor indices runs over $1, 2$ only (since the nonrelativistic spinors are of two-components), even though we kept the notation of 5-d gamma matrices on purpose to indicate possible relativistic origins of such interactions. The charge conjugation matrix $\mathcal{C}$ satisfies

$$(\gamma^{MN})^T = -\mathcal{C}\gamma^{MN}\mathcal{C}^{-1}\;, \tag{4.15}$$

and is in our convention

$$\mathcal{C} = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}\;. \tag{4.16}$$

Finally, the electromagnetic field $F^{\epsilon\epsilon'}_{kn}$ is

$$F^{\epsilon\epsilon'}_{KN} \equiv \Sigma^3_{a=1}F^a_{KN}\frac{\tau^{\epsilon\epsilon'}_a}{2}\;, \tag{4.17}$$

and, for these SU($N_F = 2$) gauge indices, τ_2 plays the same role as $\mathcal{C}$ does for the spinor indices.

Even though we are writing down a nonrelativistic action, it is important to keep in mind that there should be a fully Lorentz invariant dynamics. Once we show that the particle interaction gives rise to instanton configurations, the antiparticle interaction should follow automatically. With this in mind, let us write these terms in the honest two-component notations appropriate for the nonrelativistic spinors. With the convention of the gamma matrices (4.4), the interaction terms involving the magnetic fields, F_{ij} and F_{4i}, are

$$\begin{aligned} S_I^{\text{magnetic}} &= -\sum_s \frac{1}{2}h_s F^a_{ij}\epsilon_{ijk}\left(\mathcal{U}^{\epsilon'\epsilon_2\cdots\epsilon_{2s}}_{\beta'\alpha_2\cdots\alpha_{2s}}\right)^*\sigma_k^{\beta'\beta}\tau_a^{\epsilon'\epsilon}\mathcal{U}^{\epsilon\epsilon_2\cdots\epsilon_{2s}}_{\beta\alpha_2\cdots\alpha_{2s}} \\ &\quad -\sum_s h_s F^a_{4i}\left(\mathcal{U}^{\epsilon'\epsilon_2\cdots\epsilon_{2s}}_{\beta'\alpha_2\cdots\alpha_{2s}}\right)^*\sigma_i^{\beta'\beta}\tau_a^{\epsilon'\epsilon}\mathcal{U}^{\epsilon\epsilon_2\cdots\epsilon_{2s}}_{\beta\alpha_2\cdots\alpha_{2s}} \\ &\quad -\sum_s \frac{1}{2}k_s F^a_{ij}\epsilon_{ijk}\left(\mathcal{U}^{\epsilon_1\epsilon_2\cdots\epsilon_{2s}}_{\alpha_1\alpha_2\cdots\alpha_{2s}}\right)^*(\sigma_2\sigma_k)^{\beta\beta'}(\tau_2\tau_a)^{\epsilon\epsilon'}\mathcal{U}^{\epsilon\epsilon'\epsilon_1\cdots\epsilon_{2s}}_{\beta\beta'\alpha_1\cdots\alpha_{2s}} \\ &\quad -\sum_s k_s F^a_{4i}\left(\mathcal{U}^{\epsilon_1\epsilon_2\cdots\epsilon_{2s}}_{\alpha_1\alpha_2\cdots\alpha_{2s}}\right)^*(\sigma_2\sigma_i)^{\beta\beta'}(\tau_2\tau_a)^{\epsilon\epsilon'}\mathcal{U}^{\epsilon\epsilon'\epsilon_1\cdots\epsilon_{2s}}_{\beta\beta'\alpha_1\cdots\alpha_{2s}}\;, \end{aligned} \tag{4.18}$$

where $i, j, k = 1 \cdots 3$. With the usual t'Hooft symbol, this can be written as

$$S_I^{\text{magnetic}} = -\frac{1}{2}\sum_s h_s F^a_{mn}\bar{\eta}^b_{mn}\left(\mathcal{U}^{\epsilon'\epsilon_2\cdots\epsilon_{2s}}_{\beta'\alpha_2\cdots\alpha_{2s}}\right)^* \sigma_b^{\beta'\beta}\tau_a^{\epsilon'\epsilon}\mathcal{U}^{\epsilon\epsilon_2\cdots\epsilon_{2s}}_{\beta\alpha_2\cdots\alpha_{2s}}$$
$$-\frac{1}{2}\sum_s k_s F^a_{mn}\bar{\eta}^b_{mn}\left(\mathcal{U}^{\epsilon_1\epsilon_2\cdots\epsilon_{2s}}_{\alpha_1\alpha_2\cdots\alpha_{2s}}\right)^* (\sigma_2\sigma_b)^{\beta\beta'}(\tau_2\tau_a)^{\epsilon\epsilon'}\mathcal{U}^{\epsilon\epsilon'\epsilon_1\cdots\epsilon_{2s}}_{\beta\beta'\alpha_1\cdots\alpha_{2s}}\,. \tag{4.19}$$

The presence of the anti-self-dual 't Hooft symbol $\bar{\eta}^a_{mn}$ indicates that above interaction terms will source the smeared-out instanton field.

There will be an electric analog of these terms, S_I^{electric}, involving the electric field strengths F_{0m} instead of the magnetic field strength F_{mn}. These electric couplings cannot be derived from the soliton structure, but must be rather inferred via Lorentz invariance from the magnetic ones. Here, we chose not to display them explicitly.

4.3 Relativistic origins

As an aside, let us note that, as far as the interaction terms go, we have an obvious relativistic completion. When we proposed the nonrelativistic effective lagrangian, we implicitly assumed this underlying relativistic structure. In particular, CPT invariance is enforced. Even though the baryons are extremely heavy in the large λN_c limit, their dynamics must respect the Lorentz invariance. The difficulty involved in formulating a fully relativistic action is with the kinetic terms and constraints, rather than with interactions.

In this spirit, we note that term in S_I^{magnetic} and S_I^{electric} would follow from the following structures,

$$h'_s\bar{\Psi}^{(s)}\left(F\Psi^{(s)}\right) + f_s\bar{\Psi}^{(s)}\left(F\cdot\Psi^{(s+1)}\right)\,, \tag{4.20}$$

in terms of the relativistic spinor Ψ's. The contraction in the second terms is defined as

$$(F\cdot\Psi^{(s+1)})^{\epsilon_1\cdots\epsilon_{2s}}_{A_1A_2\cdots A_{2s}} \equiv (\tau_2 F_{MN})^{\epsilon\epsilon'}(\mathcal{C}\gamma^{MN})_{BB'}\Psi^{\epsilon\epsilon'\epsilon_1\cdots\epsilon_{2s}}_{BB'A_1A_2\cdots A_{2s}}\,, \tag{4.21}$$

which lowers the isospin and the spin representations. It is not difficult to convince oneself that these two are the only possible fermion bilinears with direct couplings to the field strength. The resulting Lorentz-covariant form of the Yang-Mills equations is

$$D^M F^{\epsilon\epsilon'}_{MN} = \cdots + \sum_s h'_s D^K\left(\gamma^{BB'}_{KN}\bar{\Psi}^{\epsilon'\epsilon_2\cdots\epsilon_{2s}}_{BA_2\cdots A_{2s}}\Psi^{\epsilon\epsilon_2\cdots\epsilon_{2s}}_{B'A_2\cdots A_{2s}}\right)$$
$$+\sum_s k_s D^K\left((\mathcal{C}\gamma_{KN})^{BB'}\tau_2^{\nu\epsilon}\bar{\Psi}^{\epsilon_1\cdots\epsilon_{2s}}_{A_1\cdots A_{2s}}\Psi^{\nu\epsilon'\epsilon_1\cdots\epsilon_{2s}}_{BB'A_1\cdots A_{2s}}\right)\,, \tag{4.22}$$

where the ellipsis denote the baryon current that account for the N_c charge of the baryon.

There are two notable differences between the relativistic and the nonrelativistic expressions. First is that we displayed in eq. (4.19) only the couplings to the magnetic field, whereas the relativistic form includes the same type of couplings to the electric field. As far as the derivation of the coupling goes, only the magnetic one can be derived since it comes from the self-dual magnetic field strength of the instanton. The electric one has to be there simply because Lorentz symmetry relates the two.

Secondly, we have h'_s in place of h_s because there is another relativistic source of the same magnetic terms. These magnetic terms are five-dimensional analog of non-anomalous magnetic moment term of four-dimensional Dirac field, and arise when we expand the minimal coupling terms in terms of nonrelativistic spinors. In [8, 10], this correction was ignored since in the large N_c limit, it represents a second order correction of order $\sim 1/m_B$. However, in the extrapolation to finite N_c, this correction, if kept, could be comparable or even larger than the leading term. This makes the extrapolation procedure somewhat ambiguous. In this note, we will stick to large N_c limit, and ignore this problem.

5. Derivation of the interaction terms

In the previous section, we speculated on possible interactions between baryons and mesons. While we discussed direct couplings to the field strengths in five dimensions, we are yet to show that the structure we gave is indeed the right one. In this section, we will generalized the work in [8, 10], and show that these couplings are inevitable consequences of the instanton origin of the baryon. As a by-product, we also compute the strength of the couplings at origin, $w = 0$.

The strategy for this goes as follows. When the instanton is quantized, its classical gauge field configuration is replaced by its expectation values as

$$F \;\rightarrow\; \langle\langle S^\dagger F S\rangle\rangle \;, \tag{5.1}$$

where $\langle\langle\cdots\rangle\rangle$ means taking expectation value on the collective coordinate wavefunctions. Componentwise, with the explicit SU(2) generators $\tau_a/2$, we have

$$F^a \;\rightarrow\; \left\langle\left\langle \mathrm{tr}\left[\tau_a S^\dagger \frac{\tau_b}{2} S\right]\right\rangle\right\rangle F^b \;, \tag{5.2}$$

so that the quantum smearing out of the instanton gauge field is determined entirely by the expectation value of the quantities,

$$\Sigma_{ab} \equiv \mathrm{tr}\left[\tau_a S^\dagger \frac{\tau_b}{2} S\right] \;. \tag{5.3}$$

If and only if any unit quanta of the baryon field can emulate such a smeared-out long range field, the effective baryon field theory would make sense.

Fortunately, the spin-isospin-locked nature of the instanton wavefunction allows us to translate this expectation value in terms of bilinears of the baryon field. The simplest of such relation was found by Adkins-Nappi-Witten for isospin 1/2 case in the context of quantized Skyrmions [20], which is

$$\langle\langle 1/2 : p', q'|\,\mathrm{tr}\left[\tau_a S^\dagger \frac{\tau_b}{2} S\right]|1/2 : p, q\rangle\rangle = -\frac{4}{3}\,\langle 1/2 : p', q'|J_a^{(\pm)} I_b|1/2 : p, q\rangle \;. \tag{5.4}$$

On the left hand side, we have overlap integrals of functions on S^3,

$$\langle\langle s : p', q'|\,\mathrm{tr}\left[\tau_a S^\dagger \frac{\tau_b}{2} S\right]|s : p, q\rangle\rangle \equiv \int_{\mathbf{S}^3} \left(D^{(s)}_{p'q'}(\xi)\right)^* D^{(s)}_{pq}(\xi)\,\Sigma_{ab}(\xi) \;, \tag{5.5}$$

whereas the quantity on the right hand side is the usual matrix elements of angular momentum operators. We can conveniently represent the right hand side as

$$4\,\langle 1/2 : p', q' | J_a^{(\pm)} I_b | 1/2 : p, q \rangle \;\equiv (\mathcal{U}(1/2 : p', q')^{\epsilon'}_{\beta'})^* \sigma_a^{\beta'\beta} \tau_b^{\epsilon'\epsilon} \mathcal{U}(1/2 : p, q)^{\epsilon}_{\beta}\;, \tag{5.6}$$

in terms of a two-component spinor field $\mathcal{U}$ in the isospin 1/2 representation.

Note that the effective action of $\mathcal{U}$ in eq. (4.19) we proposed is such that $\mathcal{U}$ bilinear sources the five-dimensional gauge field as

$$(\nabla \cdot F)^a_m \sim \nabla_n \left(\bar{\eta}^b_{nm} \mathcal{U}^\dagger (\sigma_b \tau^a) \mathcal{U} \right) + \cdots\;. \tag{5.7}$$

Thanks to the above identity (5.4), this Yang-Mills field equation implies that a $\mathcal{U}$ particle state will have a long range tail of gauge field that looks exactly like a smeared out instanton of eq. (5.2), as long as we match the precise mapping between the spinor states and the quantized instanton wavefunctions. This proves that the couplings in the previous section is indeed exactly the right ones for the spinor $\mathcal{U}$ to be interpreted as the baryon effective field for $s = 1$. In the remainder of this section, we will show how this generalizes to all isospins.

Note that together with the obligatory minimal coupling, this fixes the effective interaction of the baryon with the meson sector uniquely up to dimension six operators in five dimensions. The coupling strength is then determined by making sure the Yang-Mills solution to this equation has exactly the same size as the smeared out instanton. The subsequent reduction to four dimensions generates an infinite number of coupling constants between mesons and baryon current, as we will see shortly.

5.1 Identities for isospin-preserving processes

For $s > 1/2$, the identity (5.4) can be generalized to general integer s, as

$$\langle\langle s : p', q' | \Sigma_{ab} | s : p, q \rangle\rangle \;= -C_0(s) \langle s : p', q' | J_a I_b | s : p, q \rangle\;, \tag{5.8}$$

for arbitrary s and $-s \le p, q, p', q' \le s$ with

$$C_0(s) = \frac{1}{s(s+1)}\;. \tag{5.9}$$

To show this, let us start with the simplest case of $p = q = p' = q' = s$. For this, it is relatively easy to show that

$$\int_{\mathbf{S}^3} |D^{(s)}_{ss}|^2\, \Sigma_{ab} \;= -C_0(s) \langle s : s, s | J_a^{(\pm)} I_b | s : s, s \rangle\;, \tag{5.10}$$

holds for all 3×3 choices of (k, m) and for arbitrary integer s. The right hand side is obvious: all cases except $k = m = 3$ vanish, and for $k = m = 3$ we find $-s^2/s(s+1) = -s/(s+1)$. An explicit computation of the integral on the left hand side is also straightforward and produces the same result.

Further generalization follows from the fact that the operators on the two sides transform the same way under $\mathrm{SU}(2)_I \times \mathrm{SU}(2)_\pm$. Recalling how S transforms, we see that

$$
\begin{aligned}
\Sigma_{km} \quad \rightarrow \quad \Sigma'_{ab} &= \mathrm{tr}\left[\tau_a \left(VS^\dagger U^\dagger\right) \frac{\tau_b}{2} \left(USV^\dagger\right)\right] \\
&= \mathrm{tr}\left[(V^\dagger \tau_a V)\, S^\dagger \left(U^\dagger \frac{\tau_b}{2} U\right) S\right] \\
&= \hat{V}_a{}^c \hat{U}_b{}^d \Sigma_{cd}\,, \qquad (5.11)
\end{aligned}
$$

where $\hat{V}$ and $\hat{U}$ are the 3×3 matrix representation of V and U. Thus, Σ_{km} transform under the $\mathcal{I}$'s and $\mathcal{J}$'s exactly as the operators $J_k^{(\pm)} I_m$ would transform under I's and $J^{(\pm)}$'s. Next, consider

$$
\langle\langle s:s,s|\Sigma_{ab}|s:p,q\rangle\rangle \;=-C_0(s)\langle s:s,s|J_a I_b|s:p,q\rangle\,, \qquad (5.12)
$$

for all $-s \le p, q \le s$. It is easy to show that, of these, the only nonvanishing expressions are those with $p, q \ge s-1$. Taking $p=s, q=s-1$, for instance, we see that the left hand side reduces

$$
\begin{aligned}
\langle\langle s:s,s|\Sigma_{3+}|s:s,s-1\rangle\rangle &= \frac{1}{\sqrt{2s}}\langle\langle s:s,s|\Sigma_{3+}\mathcal{I}_-|s:s,s\rangle\rangle \\
&= \frac{1}{\sqrt{2s}}\langle\langle s:s,s|2\Sigma_{33}|s:s,s\rangle\rangle\,, \qquad (5.13)
\end{aligned}
$$

which is exactly mirrored by the left hand side because $[I_+, I_-] = 2I_3$ and $\langle s:s,s|I_- = 0$. So, the first identity in (5.12) follows from (5.10). The remaining two can be shown likewise. Continuing in this fashion, the rest of the identity in eq. (5.8) follows automatically.

The right hand side is more conveniently represented in terms of the nonrelativistic spinor of the previous section as

$$
\begin{aligned}
\langle\langle s:p',q'|\Sigma_{ab}|s:p,q\rangle\rangle &= -C_0(s)\,\mathcal{U}(s:p',q')^\dagger(J_a\otimes I_b\mathcal{U}(s:p,q)) \\
&= -C_0(s)\times s^2\times(\mathcal{U}(s:p',q')^{\epsilon'\epsilon_2\cdots\epsilon_{2s}}_{\beta'\alpha_2\cdots\alpha_{2s}})^*\sigma_a^{\beta'\beta}\tau_b^{\epsilon'\epsilon}\mathcal{U}(s:p,q)^{\epsilon\epsilon_2\cdots\epsilon_{2s}}_{\beta\alpha_2\cdots\alpha_{2s}} \\
&= -\frac{s}{s+1}\times(\mathcal{U}(s:p',q')^{\epsilon'\epsilon_2\cdots\epsilon_{2s}}_{\beta'\alpha_2\cdots\alpha_{2s}})^*\sigma_a^{\beta'\beta}\tau_b^{\epsilon'\epsilon}\mathcal{U}(s:p,q)^{\epsilon\epsilon_2\cdots\epsilon_{2s}}_{\beta\alpha_2\cdots\alpha_{2s}}, \qquad (5.14)
\end{aligned}
$$

where the operators J and I acting on $\mathcal{U}$ are of understood to be in the spin (isospin) s representation. Note that the last expression, up to an overall numerical factor, is precisely the first of the two fermion bilinears that appeared in eq. (4.19)

5.2 Generalizing to isospin-changing processes

Note that Σ_{km} are themselves spin one wavefunctions on S^3. With the non-Hermitian choice of basis $\tau_\pm = (\tau_1 \pm i\tau_2)/\sqrt{2}$, we find

$$
\begin{aligned}
\Sigma_{++}(\xi) &= \sqrt{2\pi^2/3}\, D^{(1)}_{11}(\xi) = (\xi_1 + i\xi_2)^2\,, \\
\Sigma_{3+}(\xi) &= \sqrt{2\pi^2/3}\, D^{(1)}_{01}(\xi)\,, \\
\Sigma_{+3}(\xi) &= \sqrt{2\pi^2/3}\, D^{(1)}_{10}(\xi)\,, \\
&\;\;\vdots \\
\Sigma_{--}(\xi) &= \sqrt{2\pi^2/3}\, D^{(1)}_{-1-1}(\xi) = (\xi_1 - i\xi_2)^2\,. \qquad (5.15)
\end{aligned}
$$

This means there are another set of expectation values

$$\langle\langle s:p',q'|\Sigma_{ab}|s+1:p,q\rangle\rangle\ , \tag{5.16}$$

and their complex conjugates.

For this new class, the analog of Adkins-Nappi-Witten identity are

$$\langle\langle s:p',q'|\Sigma_{ab}|s+1:p,q\rangle\rangle = -C_1(s)\ \left[\mathcal{U}(s:p',q')^\dagger\mathcal{U}(s+1:p,q)_{ab}\right]\ , \tag{5.17}$$

with

$$\begin{aligned}\mathcal{U}(s+1:p,q)_{ab} &\equiv (\widehat{\sigma_2\sigma_a})(\widehat{\tau_2\tau_b})\mathcal{U}(s+1:p,q)\ , \qquad (5.18)\\ \left((\widehat{\sigma_2\sigma_a})(\widehat{\tau_2\tau_b})\mathcal{U}(s+1:p,q)\right)^{\epsilon_1\cdots\epsilon_{2s}}_{\alpha_1\cdots\alpha_{2s}} &\equiv (\sigma_2\sigma_a)^{\beta\beta'}(\tau_2\tau_b)_{\epsilon\epsilon'}\mathcal{U}(s+1:p,q)^{\epsilon\epsilon'\epsilon_1\cdots\epsilon_{2s}}_{\beta\beta'\alpha_1\cdots\alpha_{2s}}\ ,\end{aligned}$$

defining the contracting action that reduces the spin (isospin) by one.

To show this identity and determine $C_1(s)$, it again suffices to compute the case of $p=q=s+1$. Under this restriction, the angular momentum summation rules tell us that the one and only non-vanishing matrix element on the left hand side is

$$\langle\langle s:p'=s,q'=s|\Sigma_{--}|s+1:s+1,s+1\rangle\rangle = \int_{S^3}(\xi_1-i\xi_2)^2(D^{(s)}_{ss})^*D^{(s+1)}_{s+1,s+1}\ , \tag{5.19}$$

while $p',q'<s$ producing null results. Likewise, the right hand side also vanishes except for

$$-C_1(s)\left[\mathcal{U}(s:s,s)^\dagger\left((\widehat{\sigma_2\sigma_-})(\widehat{\tau_2\tau_-})\mathcal{U}(s+1:s+1,s+1)\right)\right] = 2\times C_1(s)\ , \tag{5.20}$$

since, for any s, the only nonvanishing component of $\mathcal{U}(s:s,s)$ is

$$\mathcal{U}(s:s,s)^{11\cdots1}_{11\cdots1} = 1\ . \tag{5.21}$$

Therefore, the identity in eq. (5.17) holds for $p=q=s+1$ with

$$C_1(s) = \frac{1}{2}\int_{S^3}(x_1-ix_2)^2\left(D^{(s)}_{ss}\right)^*D^{(s+1)}_{s+1,s+1} = \frac{1}{2}\sqrt{\frac{2s+1}{2s+3}}\ . \tag{5.22}$$

Similarly with the case of the same spins, the rest of the identity would follow immediately if the contracting actions by $\tau_2\tau_m$ and $\sigma_2\sigma_k$ on $\mathcal{U}$, denoted above as $\widehat{\sigma_2\sigma_l}$, $\widehat{\tau_2\tau_l}$, themselves obeys

$$[J_a,\widehat{\sigma_2\sigma_b}] = i\epsilon_{abc}\widehat{\sigma_2\sigma_c}\ , \tag{5.23}$$

$$[I_a,\widehat{\tau_2\tau_b}] = i\epsilon_{abc}\widehat{\tau_2\tau_c}\ , \tag{5.24}$$

when acting in the space of all possible $\mathcal{U}$'s.

5.3 Strength of the magnetic couplings at origin

These generalized identities show that the couplings suggested in the previous section are indeed exactly the right ones demanded by the instanton origin of the baryons. They also fix the coefficient functions h_s's and k_s's at origin $w = 0$ unambiguously. This was done for the case of $h_{1/2}$ in [10], where it was found to be[6]

$$h_{1/2}(w = 0) = \frac{2\pi^2}{3} \frac{\rho^2}{e^2(0)} \ . \tag{5.25}$$

If we were considering the instanton soliton in R^4, this coupling would be a constant.

This is straightforwardly generalized to other h_s and k_s as follows,

$$h_s(0) = 2\pi^2 \frac{s}{s+1} \frac{\rho^2}{e^2(0)}, \qquad k_s(0) = \pi^2 \sqrt{\frac{2s+1}{2s+3}} \frac{\rho^2}{e^2(0)} \ , \tag{5.26}$$

where the factor $1/3$ in $h_{1/2}$ is replaced by $s^2 C_0(s)$ and by $C_1(s)$, respectively.

The ratio $k_{1/2}(0)/h_{1/2}(0)$ was implicit in Adkins-Nappi-Witten's consideration of two amplitudes, $\pi N \Delta$ and $\pi N N$. The ratio of the two amplitudes is directly related to the above ratio, up to normalization issues in terms of defining the amplitudes. We took care to verify that the two ratios give the same physics, which provides an independent check of our computation of the couplings.

However, the actual geometry is $R^3 \times I$ up to a nontrivial conformal factor as a function of w and this would in general imply that h_s and k_s are functions of w. Due to the fact that a stationary solution is possible only when the soliton is located at $w = 0$, we can determine these coefficient functions at $w = 0$ at best. In next section, we will finally come to the four-dimensional physics, and see how these couplings generate cubic and quartic couplings between baryons and meson in four dimensions. The fact that these coefficient functions are not well-determined away from the origin, in general, poses a systematic difficulty in predicting couplings to excited mesons beyond those associated with large N_c and large λ nature of this model. For low lying mesons, however, errors due to this are relatively well controlled.

6. Baryons interacting with mesons

So far we considered an effective field theory for the instanton soliton of a fixed size on R^{4+1}, using the approximate SO(4,1) symmetry. This effective field theory is not yet that of the four dimensional baryons in two aspects. First, even though the soliton at origin ($w = 0$) sees the approximate SO(4,1) Lorentz symmetry, a quantum of the spinor fields will see strong breaking of this away from $w = 0$. Second, we must reduce the effective field theory to four dimensions in order to identify the spinor fields with baryons of QCD. In this section, we will incorporate these two issues and produce a bona fide effective action for QCD baryons.

[6] For the proper normalization of the spinors and the coupling, it is important to recall that the convention for two-component spinors in this paper is different from that of [10] where the four-component spinor was written in terms of two related γ^5 eigenspinors. Here we are using two-component spinors which are γ^0 eigenspinors in the rest frame.

JHEP06(2008)011

6.1 Broken SO(4, 1) symmetry and a classical potential for 5D theory

The leading effect of having a nontrivial background geometry (conformally $R^{3+1} \times I$) is that the instanton soliton's mass varies with the position along the holographic direction. The leading mass comes from

$$\int_{R^3 \times I} \frac{1}{8\pi^2 e(w)^2} \operatorname{tr} F \wedge F \;, \tag{6.1}$$

and, due to the position-dependence of $1/e(w)^2$, the soliton prefers to sit near $w = 0$. If we have a relativistic formulation, this could be naturally incorporated into a position dependent mass term. For nonrelativistic formulation, the mass shows up as denominator of the quadratic spatial gradient term. Making this parameter position dependent does not seem to yield the right energetics.

As a toy model consider a spin 1/2 Dirac field with a position dependent mass

$$-i\bar{\Psi}\partial_M \gamma^M \Psi + im(x)\bar{\Psi}\Psi \;, \tag{6.2}$$

with

$$m(x) = m(s) + V(x), \qquad m(s) > 0 \text{ and } V(x) \geq 0 \;. \tag{6.3}$$

If we are interested in low momentum and low energy behavior of this field, we may as well treat V as a perturbation. The on-shell condition is then,

$$\begin{pmatrix} E & i\sigma^m p_m \\ i\bar{\sigma}^m p_m & -E \end{pmatrix} \Psi = m(s)\Psi \;, \tag{6.4}$$

with $\sigma^i = \bar{\sigma}^i$ are the usual Pauli matrices, and $\sigma_4 = i = -\bar{\sigma}_4$. Using the particle state, for which $E \simeq m(s) + O(p^2)$, and defining the two-component nonrelativistic spinors as

$$\Psi = e^{-im(s)t} \begin{pmatrix} \mathcal{U} \\ \mathcal{V} \end{pmatrix} \;, \tag{6.5}$$

the above relativistic action reduces to a non-relativistic one as

$$i\mathcal{U}^\dagger \partial_0 \mathcal{U} + \frac{1}{2m(s)} \mathcal{U}^\dagger \partial_m \partial^m \mathcal{U} - V(x)\mathcal{U}^\dagger \mathcal{U} \;. \tag{6.6}$$

This case of spin 1/2 instructs us, then, to incorporate the effect of position-dependence of $1/e(w)^2$ as a bilinear of $\mathcal{U}$ with the coefficient function,

$$V(x) = V(w) = \left(\frac{4\pi^2}{e(w)^2} - \frac{4\pi^2}{e(0)^2} \right) \tag{6.7}$$

in addition to the standard kinetic terms we have. For the baryons, this acts as a potential in the resulting Schroedinger equation for $\mathcal{U}$, which pulls the particles toward the origin $w = 0$.

The right thing to do for baryons of any isospin, therefore, is to add such a potential term to the action, so that the total action is

$$S_{5D} = S_0' + S_I^{\text{magnetic}} + S_I^{\text{electric}} \;, \tag{6.8}$$

with

$$S_0' = \int dtd^4x \left[\sum_s \left(i\mathcal{U}_s^\dagger \frac{\partial}{\partial t}\mathcal{U}_s + \sum_{m=1}^{4} \frac{1}{2m(s)}\mathcal{U}_s^\dagger(\partial_m - i\mathcal{A}_m)^2\mathcal{U}_s - V(w)\mathcal{U}_s^\dagger\mathcal{U}_s \right) \right] , \tag{6.9}$$

whereas S_I^{magnetic} is the same interaction piece as in eq. (4.19) and S_I^{electric} its electric counterpart. For most applications below, we won't need explicit form of S_I^{electric} since its form and size will be related to S_I^{magnetic} via Lorentz invariance.

6.2 Baryon-meson couplings from the dimensional reduction

Let's first consider the interactions between mesons and spin 1/2 baryons, namely nucleons. In ref. [10] the dimensional reduction was done for the relativistic theory. Here, we carry out the dimensional reduction of non-relativistic theory in 5-dimensions, and we will briefly compare the two approaches before proceeding to the higher spin cases. For spin half case, the baryonic wave functions are written as two component spinors satisfying Schrodinger equation. Specifically it satisfies

$$i\frac{\partial}{\partial t}\mathcal{U}_{1/2} = -\frac{1}{2m(s)}\left(\partial_i\partial_i + \left(\frac{\partial}{\partial w}\right)^2\right)\mathcal{U}_{1/2} + V(w)\mathcal{U}_{1/2} . \tag{6.10}$$

If we write it as a product of four-dimensional wave function and the one dimensional wave function

$$\mathcal{U}_{1/2} = B(t,x_i)_{1/2} f(w) e^{-iE_n t} , \tag{6.11}$$

where $f(w)$ satisfies the one-dimensional potential problem with energy eigenvalues E_n

$$-\frac{1}{2m(1/2)}\left(\frac{d}{dw}\right)^2 f(w) + V(w)f(w) = E_n f(w) , \tag{6.12}$$

then $B_{1/2}$ satisfies the free Schrodinger equation

$$i\frac{\partial}{\partial t}B_{1/2} = -\frac{1}{2m(1/2)}\partial_i\partial_i B_{1/2} , \tag{6.13}$$

where $i = 1\cdots 3$. We are interested in the lowest lying baryon for each isospin sector, so we will take the smallest eigenvalue E_0 and its associated ground state wavefunction $f_0(w)$.

Let us carry out the dimensional reduction of the following action describing the interactions between the spin $\frac{1}{2}$ baryons and mesons

$$\begin{aligned} S_{5D} = \int & dtd^3xdwi\mathcal{U}_{1/2}^\dagger\left(\frac{\partial}{\partial t} - i\mathcal{A}_0\right)\mathcal{U}_{1/2} + \sum_{m=1}^{4}\frac{1}{2m(1/2)}\mathcal{U}_{1/2}^\dagger(\partial_m - i\mathcal{A}_m)^2\mathcal{U}_{1/2} - V(w)\mathcal{U}_{1/2}^\dagger\mathcal{U}_{1/2} \\ & -h_{1/2}(w)\left(\mathcal{U}_{1/2}^\dagger\epsilon_{ijk}\sigma_k F_{ij}\mathcal{U}_{1/2} + 2\mathcal{U}_{1/2}^\dagger i\sigma_i F_{4i}\mathcal{U}_{1/2}\right) \\ & -\frac{1}{4e^2(w)}tr\mathcal{F}_{mn}\mathcal{F}^{mn} , \end{aligned} \tag{6.14}$$

with $F_{ij} = \sum_{a=1}^{3} \frac{1}{2} F_{ij}^a \tau^a$. Note that the minimal couplings have U(2) gauge fields while the magnetic couplings of eq. (6.14) have SU(2) gauge fields, which source SU(2) instanton fields. Recall that $\mathcal{A}_\mu(x, w)$ is expanded in terms of mesons as [5]

$$\mathcal{A}_\mu(x, w) = i\alpha_\mu(x)\psi_0(w) + i\beta_\mu(x) + \Sigma_{n\geq 1} a_\mu^{(n)}(x)\psi_{(n)}(w) \ , \tag{6.15}$$

with the gauge choice $\mathcal{A}_w = 0$. For the SU(2) part, A_μ, the first two terms may be expanded in terms of the pion and spin 1 mesons as

$$A_\mu = -\frac{2}{f_\pi}\partial_\mu \pi \psi_0(w) + \frac{i}{2f_\pi^2}[\pi, \partial_\mu \pi] + \cdots \ , \tag{6.16}$$

where $f_\pi^2 = (g_{\rm YM}^2 N_c) N_c M_{KK}^2 / 54\pi^4$ is the pion decay constant. We also need to separate U(1) part of the vector/axial-vector mesons as well. Regardless of parity, let us write

$$a_\mu^{(n)} = \begin{pmatrix} N_c/2 & 0 \\ 0 & N_c/2 \end{pmatrix} \omega_\mu^{(n)} + v_\mu^{(n)} \ , \tag{6.17}$$

so the w's are the isosinglets and v's isotriplets.

In the large λN_c limit, the baryon wave function is sharply peaked around $w = 0$. When we integrate over w-direction, $|f_0(w)|^2$ may be approximated as a delta function, e.g., $\int dw |f_0(w)|^2 \psi_{(n)}(w) = \psi_{(n)}(0)$. One of the consequence is that the dimensional reduction of A_i^2 can be written as the product of that of A_i, i.e.,

$$\begin{aligned}
&\int dt\, d^3x\, dw \sum_{i=1}^{3} \mathcal{U}_{1/2}^\dagger (\mathcal{A}_i)^2 \mathcal{U}_{1/2} \\
&\qquad = \int dt\, d^3x B_{1/2}^\dagger \left(\alpha_i(t,x)\psi_0(0) + \beta_i(t,x) - i\Sigma_{n\geq 1} a_i^{(n)}(t,x)\psi_{(n)}(0) \right)^2 B_{1/2} \\
&\qquad = \int dt\, d^3x B_{1/2}^\dagger \left(\beta_i(t,x) - i\Sigma_{n\geq 0} a_i^{(2n+1)}(t,x)\psi_{(2n+1)}(0) \right)^2 B_{1/2} \ ,
\end{aligned} \tag{6.18}$$

where we use that $\psi_{(2n)}(w), \psi_{(2n+1)}(w)$ are an odd and even function of w respectively.

The dimensional reduction of the minimal coupling produce vector-like couplings as those of V_μ

$$S_{\rm minimal} = \int dt\, d^3x \left(iB_{1/2}^\dagger (\frac{\partial}{\partial t} - iV_0) B_{1/2} + \sum_{i=1}^{3} \frac{1}{2m(1/2)} B_{1/2}^\dagger (\partial_i - iV_i)^2 B_{1/2} \right) ., \tag{6.19}$$

where V_μ collects all the vector mesons (as opposed to axial vector mesons) in their hidden local gauge symmetry [19, 5] form

$$V_\mu \equiv i\beta_\mu(t,x) + \Sigma_{n\geq 0} a_\mu^{(2n+1)}(t,x)\psi_{(2n+1)}(0), \qquad \mu = 0 \cdots 3 \ . \tag{6.20}$$

The coupling constants for couplings to $\beta_\mu, \omega_\mu^{(2n+1)}, v_\mu^{(2n+1)}$ are 1, $N_c\psi_{(2n+1)}(0), \psi_{(2n+1)}(0)$ respectively . Similarly one can derive the interaction terms coming from five-dimensional magnetic couplings, which generates the leading couplings to isotriplet axial vector mesons, $v^{(2n)}$, as well as derivative couplings to isotriplet vector mesons, $v^{(2n+1)}$,

$$
\begin{aligned}
S_{\rm axial} = & -\int dt\, d^3x h_{1/2}(0) 2B^\dagger_{1/2}\sigma_i \left(\frac{4i}{\pi}\alpha_i + \sum_{n\geq 1} v_i^{(2n)} \psi'_{(2n)}(0) \right) B_{1/2} \\
& + h_{1/2}(0) B^\dagger_{1/2}\epsilon_{ijk}\sigma_k \sum_{n\geq 0} \left(\partial_i v_j^{(2n+1)} - \partial_j v_i^{(2n+1)} \right) B_{1/2}\ ,
\end{aligned}
\tag{6.21}
$$

where $\psi' = d\psi/dw$ with $\psi'_0(0) = 4/\pi$. The axial coupling strength is

$$
h_{1/2}(0) = \frac{2\pi^2}{3}\frac{\rho^2}{e^2(0)} = \sqrt{\frac{1}{30}}\frac{N_c}{M_{KK}}\ . \tag{6.22}
$$

One can see that the interaction terms for $\alpha_\mu, v_\mu^{(2n)}$ arise only from the 5-dimensional magnetic couplings, which is observed in relativistic case in the large N_c limit.

It is straightforward to generalize to baryons with general spins. All of the wavefunctions are sharply peaked around w and the overlap integrals in w direction act as delta function, which is true for the overlap integral for wavefunctions of different spins. The minimal terms are given by

$$
S_{\rm minimal} = \int dt\, d^3x \sum_s \left(iB_s^\dagger(\frac{\partial}{\partial t} - iV_0)B_s + \sum_{i=1}^{3} \frac{1}{2m_s} B_s^\dagger(\partial_i - iV_i)^2 B_s \right)\ , \tag{6.23}
$$

where $U_s = U_s(t,x)$ denotes the four-dimensional wave function of the baryon with spin s from now on. With $v_\mu^{(n)} \equiv \sum_{a=1}^3 v_\mu^{(n)\ a}\tau^a/2$, the contribution from magnetic terms (4.18) are given by

$$
\begin{aligned}
S_{\rm axial} = & \int dt d^3x - \sum_s \frac{1}{2} h_s(0) \sum_{n\geq 0} \left(\partial_i v_j^{(2n+1)} - \partial_j v_i^{(2n+1)} \right)^a \epsilon_{ijk} \left(B^{\epsilon'\epsilon_2\cdots\epsilon_{2s}}_{s;\ \beta'\alpha_2\cdots\alpha_{2s}} \right)^* \sigma_k^{\beta'\beta} \tau_a^{\epsilon'\epsilon} B^{\epsilon\epsilon_2\cdots\epsilon_{2s}}_{s;\ \beta\alpha_2\cdots\alpha_{2s}} \\
& - \sum_s h_s(0) \left(\frac{4i}{\pi}\alpha_i^a + \sum_{n\geq 1} v_i^{(2n)\ a} \psi'_{(2n)}(0) \right) \left(B^{\epsilon'\epsilon_2\cdots\epsilon_{2s}}_{s;\ \beta'\alpha_2\cdots\alpha_{2s}} \right)^* \sigma_i^{\beta'\beta} \tau_a^{\epsilon'\epsilon} B^{\epsilon\epsilon_2\cdots\epsilon_{2s}}_{s;\ \beta\alpha_2\cdots\alpha_{2s}} \\
& - \sum_s \frac{1}{2} k_s(0) \sum_{n\geq 0} \left(\partial_i v_j^{(2n+1)} - \partial_j v_i^{(2n+1)} \right)^a \epsilon_{ijk} (B^{\epsilon_1\epsilon_2\cdots\epsilon_{2s}}_{s;\ \alpha_1\alpha_2\cdots\alpha_{2s}})^* (\sigma_2\sigma_k)^{\beta\beta'} (\tau_2\tau_a)^{\epsilon\epsilon'} B^{\epsilon\epsilon'\epsilon_1\cdots\epsilon_{2s}}_{s+1;\ \beta\beta'\alpha_1\cdots\alpha_{2s}} \\
& - \sum_s k_s(0) \left(\frac{4i}{\pi}\alpha_i^a + \sum_{n\geq 1} v_i^{(2n)\ a} \psi'_{(2n)}(0) \right) (B^{\epsilon_1\epsilon_2\cdots\epsilon_{2s}}_{s;\ \alpha_1\alpha_2\cdots\alpha_{2s}})^* (\sigma_2\sigma_i)^{\beta\beta'} (\tau_2\tau_a)^{\epsilon\epsilon'} B^{\epsilon\epsilon'\epsilon_1\cdots\epsilon_{2s}}_{s+1;\ \beta\beta'\alpha_1\cdots\alpha_{2s}}
\end{aligned}
\tag{6.24}
$$

where

$$
\begin{aligned}
h_s(0) &= 2\pi^2 \frac{s}{s+1}\frac{\rho^2}{e^2(0)} = \frac{\sqrt{6}}{2\sqrt{5}}\frac{s}{s+1}\frac{N_c}{M_{KK}}\ , \\
k_s(0) &= \pi^2 \sqrt{\frac{2s+1}{2s+3}}\frac{\rho^2}{e^2(0)} = \frac{\sqrt{6}}{4\sqrt{5}}\sqrt{\frac{2s+1}{2s+3}}\frac{N_c}{M_{KK}}\ .
\end{aligned}
\tag{6.25}
$$

6.3 Baryon-pion interactions

These two sets of interaction terms contain couplings to the pion field $\pi(x)$ from

$$A^a_\mu = -\frac{2}{f_\pi}\partial_\mu \pi^a \psi_0(0) - \frac{1}{2f_\pi^2}\varepsilon^{abc}\pi^b\partial_\mu\pi^c + \cdots . \tag{6.26}$$

From S_{minimal}, we find

$$\begin{aligned}
&\sum_s B_s^\dagger \left(-\frac{1}{2f_\pi^2}\varepsilon^{abc}\pi^b\partial_0\pi^c\frac{\tau^a}{2}\right)B_s \\
&+\frac{1}{2m(s)}\sum_{i=1}^{3} B_s^\dagger\left(\frac{i}{f_\pi^2}\varepsilon^{abc}\pi^b\partial_i\pi^c\partial_i + \frac{i}{2f_\pi^2}\varepsilon^{abc}\pi^b\partial_i\partial_i\pi^c\right)\frac{\tau^a}{2}B_s \\
&-\frac{1}{4f_\pi^4}B_s^\dagger\left(\epsilon_{abc}\pi^b\partial_i\pi^c\frac{\tau^a}{2}\epsilon_{\text{def}}\pi^e\partial_i\pi^f\frac{\tau^d}{2}\right)B_s + \cdots ,
\end{aligned} \tag{6.27}$$

up to terms higher order in $1/f_\pi$, where the gauge generators τ_a's act only on the first gauge doublet index of B_s's, and, from S_{axial}

$$\begin{aligned}
&\sum_s h_s(0)\frac{8}{\pi f_\pi}\partial_i\pi^a \;\left(B^{\epsilon'\epsilon_2\cdots\epsilon_{2s}}_{s;\;\beta'\alpha_2\cdots\alpha_{2s}}\right)^*\sigma_i^{\beta'\beta}\tau_a^{\epsilon'\epsilon}B^{\epsilon\epsilon_2\cdots\epsilon_{2s}}_{s;\;\beta\alpha_2\cdots\alpha_{2s}} \\
&+\sum_s h_s(0)\frac{1}{2f_\pi^2}\varepsilon^{abc}\partial_i\pi^b\partial_j\pi^c\epsilon_{ijk}\left(B^{\epsilon'\epsilon_2\cdots\epsilon_{2s}}_{s;\;\beta'\alpha_2\cdots\alpha_{2s}}\right)^*\sigma_k^{\beta'\beta}\tau_a^{\epsilon'\epsilon}B^{\epsilon\epsilon_2\cdots\epsilon_{2s}}_{s;\;\beta\alpha_2\cdots\alpha_{2s}} \\
&+\sum_s k_s(0)\frac{8}{\pi f_\pi}\partial_i\pi^a\left(B^{\epsilon_1\epsilon_2\cdots\epsilon_{2s}}_{s;\;\alpha_1\alpha_2\cdots\alpha_{2s}}\right)^*(\sigma_2\sigma_i)^{\beta\beta'}(\tau_2\tau_a)^{\epsilon\epsilon'}B^{\epsilon\epsilon'\epsilon_1\cdots\epsilon_{2s}}_{s+1;\;\beta\beta'\alpha_1\cdots\alpha_{2s}} \\
&+\sum_s k_s(0)\frac{1}{2f_\pi^2}\varepsilon^{abc}\partial_i\pi^b\partial_j\pi^c\epsilon_{ijk}\left(B^{\epsilon_1\epsilon_2\cdots\epsilon_{2s}}_{s;\;\alpha_1\alpha_2\cdots\alpha_{2s}}\right)^*(\sigma_2\sigma_k)^{\beta\beta'}(\tau_2\tau_a)^{\epsilon\epsilon'}B^{\epsilon\epsilon'\epsilon_1\cdots\epsilon_{2s}}_{s+1;\;\beta\beta'\alpha_1\cdots\alpha_{2s}} \\
&+\cdots ,
\end{aligned} \tag{6.28}$$

again up to terms higher order in $1/f_\pi$. These are generalization of pion-nucleon couplings, which altogether may be written compactly as

$$\begin{aligned}
&h_{1/2}(0)\frac{16}{\pi f_\pi}B_{1/2}^\dagger\sigma_i\partial_i\pi B_{1/2} - h_{1/2}(0)B^\dagger_{1/2}\epsilon_{ijk}\sigma_k\frac{i}{2f_\pi^2}[\partial_i\pi,\partial_j\pi]B_{1/2} \\
&+\frac{i}{2f_\pi^2}B^\dagger_{1/2}[\pi,\partial_0\pi]B_{1/2} + \frac{1}{2m(1/2)}\frac{1}{2f_\pi^2}B^\dagger_{1/2}([\pi,\partial_i\partial_i\pi]+2[\pi,\partial_i\pi]\partial_i)B_{1/2} \\
&+\frac{1}{2m_0}B^\dagger_{1/2}\left(\frac{1}{4f_\pi^4}[\pi,\partial_i\pi][\pi,\partial_i\pi]\right)B_{1/2} + \cdots ,
\end{aligned} \tag{6.29}$$

similarly in the $1/f_\pi$ expansion.

6.4 A comment on subleading corrections and relativistic formulation

One major difference between the relativistic and the nonrelativistic approaches is a loss, or ambiguity, of subleading terms. A good illustration of this is the leading axial coupling to the pion. In the large N_c limit, the magnetic coupling gives the dominant contribution scaling linearly with N_c. The relativistic kinetic term, however, also contribute $O(1)$ term

JHEP06(2008)011

inversely proportional to the mass of the baryon. The mechanism behind the latter is precisely the same as how one obtains $g = 2$ nonanomalous gyromagnetic ratio from the minimal coupling of a Dirac fermion to electromagnetic gauge field.

Once we abandon the relativistic formulation, therefore, such terms can only be included in the interaction terms somewhat arbitrarily. Just as one cannot predict $g = 2$ from Schroedinger equation, we cannot compute the subleading term to pion-baryon coupling due to the minimal coupling. This problem is not confined to the pion coupling and is applicable to all terms we are considering. Obviously this does not affect our leading contributions, but it also tells us that finding a fully relativistic form is essential for improving the present result to next order. We should note that this sort of problem also manifest itself in computation of four-dimensional mass of the baryons.

7. Summary

In this paper, we generalize the derivation of the interactions between mesons and nucleons carried out in ref. [8, 10] to the interactions between mesons and baryons of arbitrary half-integer spins for the two-flavor case ($N_F = 2$). Following the approach given in ref. [8], we resort to the instantonic origin of the baryon fields, and produced general prescriptions and formulae that determine the strength of each interaction term in the large N_c and large λ limit. For the nucleon case (isospin 1/2), this program has produced a relativistic action and a rich phenomenology [8, 10, 14], improving the Skyrme model computations by Adkins, Nappi and Witten [20] substantially.

Baryons are realized holographically as small instanton solitons in five dimensions with a Coulombic hair, whose quantization gives rise to baryons of (half-)integer spins. The corresponding on-shell field content may be realized as fermionic fields with symmetric spinor indices under the little group as well as the same number of symmetric isospin indices under the flavor group. However it's not clear how to write down the relativistic action since the relativistic version of such multi-spinor fermion is not known. The main difficulty is in finding a relativistic formulation where the appropriate constraints may be built in at the level of action. Due to this technical difficulty, we chose to consider the non-relativistic limit for the baryons instead. This limit is sufficient, as it turned out, if we look only for the leading large N_c results.

Since the spin fields arise from the quantization of the instanton, the interaction terms between the holographic baryons and the five-dimensional flavor gauge fields should be compatible with the semiclassical instantonic configuration. Out of this consideration, a single term, called the magnetic term, together with the usual minimal coupling essentially determines all the interactions between the mesons and baryons upon the dimensional reduction of the five dimensional nonrelativistic actions down to four dimensions because the dimensional reduction of U(2) gauge field give rise to towers of mesons including pion fields. In particular, when restricted to the sector of nucleons, the nonrelativistic approach adopted here reproduces the same results as in ref. [10] derived from the relativistic case in the large N_c limit .

For subleading corrections in $1/N_c$, which would be relevant for $N_c = 3$ case as appropriate for real QCD, some ambiguities remain in part because we had to use nonrelativistic formulation and also in part because of other more fundamental reasons. These include other $1/N_c$ corrections (notably the one due to quenching and also due to the inherent limitations present in any AdS/QCD models) which are not well understood either.

We hope that this work will provide the starting point for comparing with the experimental data or other field theoretical computations on the interactions between mesons and baryons. Finally all of the baryons we consider have just SU(2) isospin symmetry. It would be interesting to extend the current work to SU(3) case, which would include strange baryons.

Acknowledgments

We thank Ho-Ung Yee for a usueful comment on the manuscript. This work is supported in part by the Science Research Center Program of KOSEF through the Center for Quantum Space-Time (CQUeST) of Sogang-University with the grant number R11-2005-021 (J.P.,P.Y.), by the Postech BSRI research fund 2007 (J.P.), and by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD, Basic Research Promotion Fund, KRF-2007-314-C00052) (P.Y.). Both authors are also supported by the Stanford Institute for Theoretical Physics (SITP Quantum Gravity visitor fund).

References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [hep-th/9711200]; S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109]; E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[2] E. Witten, *Anti-de Sitter space, thermal phase transition and confinement in gauge theories*, *Adv. Theor. Math. Phys.* **2** (1998) 505 [hep-th/9803131].

[3] C. Csáki, H. Ooguri, Y. Oz and J. Terning, *Glueball mass spectrum from supergravity*, *JHEP* **01** (1999) 017 [hep-th/9806021].

[4] R.C. Brower, S.D. Mathur and C.-I. Tan, *Glueball spectrum for QCD from AdS supergravity duality*, *Nucl. Phys.* **B 587** (2000) 249 [hep-th/0003115].

[5] T. Sakai and S. Sugimoto, *Low energy hadron physics in holographic QCD*, *Prog. Theor. Phys.* **113** (2005) 843 [hep-th/0412141]; *More on a holographic dual of QCD*, *Prog. Theor. Phys.* **114** (2006) 1083 [hep-th/0507073].

[6] T.H.R. Skyrme, *A unified field theory of mesons and baryons*, *Nucl. Phys.* **31** (1962) 556.

[7] K. Hashimoto, C.-I. Tan and S. Terashima, *Glueball decay in holographic QCD*, *Phys. Rev.* **D 77** (2008) 086001 [arXiv:0709.2208].

[8] D.K. Hong, M. Rho, H.-U. Yee and P. Yi, *Chiral dynamics of baryons from string theory*, *Phys. Rev.* **D 76** (2007) 061901 [hep-th/0701276].

[9] H. Hata, T. Sakai, S. Sugimoto and S. Yamato, *Baryons from instantons in holographic QCD*, hep-th/0701280.

[10] D.K. Hong, M. Rho, H.-U. Yee and P. Yi, *Dynamics of baryons from string theory and vector dominance*, *JHEP* **09** (2007) 063 [arXiv:0705.2632].

[11] K.-Y. Kim, S.-J. Sin and I. Zahed, *Dense and hot holographic QCD: finite baryonic E field*, arXiv:0803.0318.

[12] O. Bergman, G. Lifschytz and M. Lippert, *Response of holographic QCD to electric and magnetic fields*, *JHEP* **05** (2008) 007 [arXiv:0802.3720].

[13] K.-Y. Kim, S.-J. Sin and I. Zahed, *The chiral model of Sakai-Sugimoto at finite baryon density*, *JHEP* **01** (2008) 002 [arXiv:0708.1469].

[14] D.K. Hong, M. Rho, H.-U. Yee and P. Yi, *Nucleon form factors and hidden symmetry in holographic QCD*, *Phys. Rev.* **D 77** (2008) 014030 [arXiv:0710.4615].

[15] E. Witten, *Baryons and branes in anti de Sitter space*, *JHEP* **07** (1998) 006 [hep-th/9805112].

[16] M.F. Atiyah and N.S. Manton, *Skyrmions from instantons*, *Phys. Lett.* **B 222** (1989) 438.

[17] T. Fujiwara et al, *An effective Lagrangian for pions, ρ mesons and skyrmions*, *Prog. Theor. Phys.* **74** (1985) 128;
U.-G. Meissner, N. Kaiser, A. Wirzba and W. Weise, *Skyrmions with ρ and ω mesons as dynamical gauge bosons*, *Phys. Rev. Lett.* **57** (1986) 1676;
U.G. Meissner and I. Zahed, *Skyrmions in the presence of vector mesons*, *Phys. Rev. Lett.* **56** (1986) 1035.

[18] K. Nawa, H. Suganuma and T. Kojo, *Brane-induced Skyrmions: baryons in holographic QCD*, *Prog. Theor. Phys. Suppl.* **168** (2007) 231 [hep-th/0701007].

[19] M. Bando, T. Kugo and K. Yamawaki, *Nonlinear realization and hidden local symmetries*, *Phys. Rept.* **164** (1988) 217.

[20] G.S. Adkins, C.R. Nappi and E. Witten, *Static properties of nucleons in the Skyrme model*, *Nucl. Phys.* **B 228** (1983) 552.

[21] E. Witten, *Current algebra, baryons and quark confinement*, *Nucl. Phys.* **B 223** (1983) 433.

[22] D. Finkelstein and J. Rubinstein, *Connection between spin, statistics, and kinks*, *J. Math. Phys.* **9** (1968) 1762.

JHEP06(2008)011